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# RESTRICTED POSITIONAL CONTROL OF A LARGE DYNAMICAL SYSTEM†

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A positional control, restricted in modulus, which steers a large system from a bounded domain of phase space into a specified neighbourhood of the origin of coordinates is proposed. The sufficient conditions that pick out the set of systems which allow this transition are obtained; the necessary time of motion is estimated, and numerical examples are given. This paper continues the investigation of positional control laws for large dynamical systems under the action of geometrically constrained control described in [1, 2] and is related to [3–6].

THE DYNAMICAL system under the action of a control restricted in magnitude

$$x^{\cdot} = Fx + Gu, \quad |u| \leqslant 1, \quad x \in R^n, \quad u \in R^m \tag{1}$$

is considered. Here $F$ and $G$ are the corresponding matrices of the phase state and control. They satisfy the condition of complete controllability [7]

$$\operatorname{rank} \| G, FG, F^2 G, \ldots, F^{n-1} G \| = n \tag{2}$$

Let the bounded domain of acceptable initial positions of system (1)

$$| x_0 | \leqslant R_1, \quad R_1 = \text{const} > 0. \tag{3}$$

be specified.

It is required to design a positional control $u(x, t)$, restricted in magnitude $|u| \leqslant 1$, such that it steers the system from any point of domain (3) into a specified neighbourhood

$$x^{\mathrm{T}} S_{\mathrm{T}} x \leqslant R_2^2, \quad R_2 = \text{const} > 0 \tag{4}$$

of the origin of coordinates, $S_{\mathrm{T}}$ being a specified positive definite matrix.

Assume that domain (3) is not contained in domain (4). This is equivalent to the inequality

$$R_1^2 > R_2^2 \| S_{\mathrm{T}} \|^{-1} \tag{5}$$

We will consider two versions of the positional control

$$\begin{aligned} &u_i(x, t) = k_i(t)\, G^{\mathrm{T}} S_i(t)\, x \\ &k_i(t) = -R_i^{-1} \| G^{\mathrm{T}} \|^{-1} \| S_i(t) \|^{-1/i} \end{aligned} \tag{6}$$

Here $k_i(t)$ $(i = 1, 2)$ are scalar functions of time, and $\|G^{\mathrm{T}}(t)\|$ and $\|S_i(t)\|$ are Euclidean norms of the matrices $G^{\mathrm{T}}$ and $S_i(t)$.

Let the matrix $S_i(t)$ in the control law (6) be specified by the condition

$$(x^{\mathrm{T}} S_i(t)\, x)^{\cdot} = 0, \quad S_i(T) = S_{\mathrm{T}} \tag{7}$$

The derivative with respect to time is calculated using the system obtained after substituting control (6) into equation of motion (1). Then

$$x^{\mathrm{T}} \Phi_i(S^{\cdot}_i, S_i) x = 0 \tag{8}$$

$$\Phi_i(S^{\cdot}_i, S_i) \equiv S_i^{\cdot} + 2k_i(t)\, S_i G G^{\mathrm{T}} S_i + S_i F + F^{\mathrm{T}} S_i$$

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Equality (8) holds for any $x$ and hence the matrix $S_i$ satisfies the equation with the boundary condition

$$\Phi_i(S_i, S_i) = 0, \quad S_i(T) = S_T \tag{9}$$

From relation (7) it follows that system (1) will pass, under the action of control (6), from any position $x_0$ belonging to the domain

$$x_0^T S_i(t) x_0 \leqslant R_2^2 \tag{10}$$

to the desired position (4) in a time $T - t_i$.

*Lemma 1.* Let $R_1$ (3), $R_2$ (4) and the matrix $S_i(t_i)$ be related by the condition

$$|x_0|^2 \leqslant R_1^2 = \| S_i(t_i)\|^{-1} R_2^2 \tag{11}$$

The control $u_i(x, t)$ being realized on trajectories then admits of the estimate

$$\begin{gathered} |u_1(x, t)| \leqslant \max_t \| S_1(t_1)\|^{1/2} \| S_1(t)\|^{-1/2} \\ |u_2(x, t)| \leqslant 1, \quad t \in [t_i, T], \quad i = 1, 2 \end{gathered} \tag{12}$$

and guarantees the transition of system (1) from any point of domain (3) into the specified neighbourhood (4) of the origin of coordinates in a time $T - t_i$.

*Proof.* We will estimate the value of the control $u_i(x, t)$ on the trajectories which start in domain (10). The definitions of controls (6) imply the inequality

$$|u_i(x, t)| \leqslant \max_{x,t} \| G^T S_i(t) x \| / (R_i \| G^T \| \| S_i \|^{1/i}) \tag{13}$$

The maximum is calculated over all values of $x$ and $t$ which satisfy the conditions $x^T S_i(t) x \leqslant R_2^2$ and $T \geqslant t \geqslant t_i$. One can prove that

$$\max_{x} \| S_i(t) x \| = \| S_i(t)\|^{1/2} R_2 \quad \text{for} \quad x^T S_i(t) x \leqslant R_2^2$$

If we substitute this expression for $i = 2$ into inequality (13), we obtain estimate (12) of the magnitude of the control $u_2(x, t)$ on the trajectories starting in domain (10). Similarly, we have the estimate

$$|u_1(x, t)| \leqslant \max_t (R_2 / (\| S_1(t)\|^{1/2} R_1)), \quad t_1 \leqslant t \leqslant T$$

for $u_1(x, t)$.

Estimate (12) follows from this inequality and relation (11); the latter implies that domain (3) belongs to domain (10).

Equation (7) and condition (11) imply the sequence of relations

$$(x^T S_T x) = (x_0^T S_i(t_i) x_0) \leqslant |x_0|^2 \| S_i(t_i)\| \leqslant R_1^2 \| S_i(t_i)\| = R_2^2$$

Condition (4) is satisfied, which proves the second assertion of the lemma.

Let us write the equation which specifies $\|S_i(t)\|$. We put $S_i = V_i^{-1}$ in Eq. (9). Then the equality $S_i^\bullet = V_i^{-1} V_i V_i^{-1}$ holds. As a result, we obtain the equation and the boundary condition which are satisfied by the matrix function

$$\begin{gathered} V_i(t) = \exp(F(t - T))(S_T^{-1} + R_i^{-1} \int_0^{T-t} I(\xi) \mu_i^{1/i}(\xi) d\xi) \exp(F^T(t - T)) \\ I(\xi) \equiv 2 \| G^T \|^{-1} \exp(F\xi) G G^T \exp(F^T \xi) \\ \mu_i(\xi) \equiv \| S_i(T - \xi)\|^{-1} \equiv \min_l l^T V_i(T - \xi) l, \quad |l| = 1 \end{gathered} \tag{14}$$

This may be verified by substitution.

Let us consider the "inverse" time $\tau = T - t$. Using (14) we obtain the integral equation for $\mu_i(\tau)$

$$\mu_i(\tau) = \min_l l^T W_i(\tau, \mu_i^{1/i}(\cdot), R_i) l \tag{15}$$

where $W_i(\tau, \mu_i^{1/i}(\cdot), R_i$ is the right-hand side of relation (14), and the dot in the expression $\mu_i(\cdot)$ denotes the fact that $W_i$ depends on values of $\mu_i(\xi)$ in the interval $(0, \tau)$.

We will estimate the rate of increase of $\mu_i(\tau)$ as $\tau \to \infty$. To do this, we will write $\tau$ in the form $\tau = N\Delta + \epsilon$ where $0 \leqslant \epsilon \leqslant \Delta$, $\Delta = \text{const} > 0$ and $N$ is an integer. We write the integral on the right-hand side of Eq. (15) as the sum of $N$ integrals and introduce the new variable $\xi$ of integration by the formula $\xi = j\Delta + \zeta$, $0 \leqslant \zeta < \Delta$, $j = 0, 1, 2, \ldots$ We then have

$$\mu_1(\tau) = \min_l l^{\mathrm{T}} \exp(-F\tau)\,[S_{\mathrm{T}}^{-1} + R_1^{-1} \sum_{j=0}^{N-1} \int_0^{\Delta} I(j\Delta+\zeta)\,\mu_1(j\Delta+\zeta)\,d\zeta + \tag{16}$$

$$+ R_1^{-1} \int_0^{\epsilon} I(N\Delta+\zeta)\,\mu_1(N\Delta+\zeta)\,d\zeta]\exp(-F^{\mathrm{T}}\tau)\,l,\quad |l| = 1$$

We will find a lower bound for $\inf_\zeta \mu_1(j\Delta+\zeta)$, $\zeta \in [0, \Delta)$. To do this, we consider $N$ values of $\mu_{ij}$ ($j = 0, 1, \ldots, N-1$) specified recurrently below such that

$$0 < \mu_{1j} \leqslant \inf_\zeta \mu_1(j\Delta+\zeta),\quad 0 \leqslant \zeta < \Delta$$

From Eq. (16) and the definition of $\mu_{ij}$ we have

$$\mu_1(\tau) \geqslant \mu_{1N}(\tau),\quad \tau \in [N\Delta, (N+1)\Delta)$$

$$\mu_{1N}(\tau) \equiv \| \exp(F\tau) \|^{-2}\, \| S_{\mathrm{T}} \|^{-1} + C_0 \sum_{j_0=0}^{N-1} \mu_{1j}\, \| \exp(F(\tau - j\Delta)) \|^{-2} \tag{17}$$

$$C_0 = R_1^{-1} \min_l l^{\mathrm{T}} \int_0^{\Delta} I(\xi)\,d\xi\, l,\quad |l| = 1$$

where $C_0 > 0$ by virtue of condition (2) of complete controllability [7]. Assume that the eigenvalues $\lambda_k$ of the matrix $F(1)$ with the largest real parts $\alpha$ have simple elementary divisors (assumption A). Then for $\tau \geqslant \tau_j$, the limit [8]

$$\| \exp(F(\tau - \tau_j)) \| \leqslant C \exp \alpha(\tau - \tau_j),\quad \tau \geqslant \tau_j,\quad C > 0 \tag{18}$$

holds.

Suppose that the homogeneous system (1) is stable. Due to the stability of the system, either assumption A is valid and $\alpha = 0$ or the condition $\alpha < 0$ holds [8]. In any case, a limit of the form (18) holds, and the minimum of the right-hand side of the inequality which is obtained by substituting this estimate into relation (17) is attained at $\tau = N\Delta$. Thus, we obtain the recurrent relation

$$\mu_1(\tau) \geqslant \mu_{1N} = C_1 \exp(-2\alpha\Delta N) +$$

$$+ C_2 \sum_{j=0}^{N-1} \mu_{1j} \exp(2\alpha\Delta(j - N)),\quad \tau \in [N\Delta, (N+1)\Delta) \tag{19}$$

$$C_1 = C^{-2} \| S_{\mathrm{T}} \|^{-1} > 0,\quad C_2 = C_0 C^{-2} > 0$$

Equation (19) has the solution

$$\mu_{10} = C_1,\quad \mu_{1N} = C_1 \exp\rho\,(\exp\rho + C_2)^{N-1}$$
$$N \geqslant 1,\quad \rho = -2\alpha\Delta \geqslant 0 \tag{20}$$

It is impossible to sum the relations for $\mu_{2N}$ ($N = 1, 2, 3, \ldots$) similar to the recurrent equations (19), because for $i = 2$ the quantity $\mu_2(\xi)$ occurs non-linearly on the right-hand side of Eqs (15). But we note the following fact. Let the instants $\tau_i = T - t_i$ ($i = 1, 2$) be the minimum positive roots of the following equations

$$\mu_i(\tau_i) = \| S_i(t_i) \|^{-1} = R_1^2 R_2^{-2} \tag{21}$$

which ensure that conditions (11) are satisfied. The estimate $\mu_1(\tau_1) \geqslant \max_\tau \mu_1(\tau)$, $\tau_1 \geqslant \tau \geqslant 0$ and, hence, the estimate of the magnitude of control (1) follows from the fact that the root $\tau_1$ is a minimum and inequality (5) holds.

Equalities (21) and the relation $\mu_1^{1/2}(\xi) < R_1 R_2^{-1}$ for $\tau_1 > \xi \geqslant 0$ ensure the limit $\mu_1(\xi) R_1^{-1} < \mu_1^{1/2}(\xi) R_2^{-1}$. This fact and Eqs (15) for $i = 1, 2$ imply the inequality $\mu_2(\tau) > \mu_1(\tau)$ for $\tau_2 \geqslant \tau > 0$, and, as a consequence, the limit

$$\tau_1 > \tau_2 \tag{22}$$

holds.

The limits (19) and (20) of the rate of increase in $\mu_1(\tau)$ as $\tau \to \infty$ and inequality (22) imply the following lemma.

*Lemma 2.* Let system (1) be completely controllable and let the corresponding homogeneous system be stable. Then instants $t_j$ ($j = 1, 2, t_1 < t_2 < T$) exist which satisfy relations (21).

The next theorem follows from Lemmas 1 and 2.

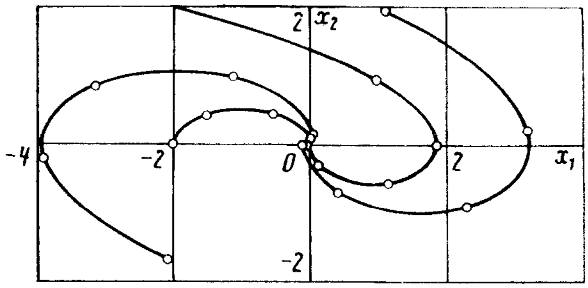


FIG. 1.

*Theorem.* Suppose system (1) is completely controllable and the corresponding homogeneous system is stable. Then instants $t_i$ $(i = 1, 2)$ exist for any point of domain (3) of the initial positions of system (1) which satisfy relation (21) and are such that controls (6) $u_i(x, t)$ $(i = 1, 2, t \in [t_i, T])$ restricted in magnitude, namely, $|u_i(x, t)| \leq 1$, steer system (1) from any point of domain (3) to the specified neighbourhood (4) of the origin of coordinates in a time $\tau_i = T - t_i$, $\tau_1 > \tau_2$, respectively.

*Remark.* The conditions of Lemma 2 guarantee the existence of the instants $t_i$ satisfying relations (21). If the existence of these instants can be proved in another way, for instance, numerically, then all the statements of the theorem will also be satisfied.

*Examples.* Equation (15) was solved numerically using a "predictor-corrector" difference scheme of the second order of accuracy. The integration was carried out up to the first instant $\tau_i$ when conditions (21) are met. The matrix $S_i(t) = V_i^{-1}(t)$ was computed from Eq. (14) simultaneously when solving Eq. (15). Then the motion of the system was simulated using the control law (6) and the matrix $S_i$. The results of computations showed that the control law (6) for $i = 2$, is much more effective in speed of response than the same law for $i = 1$. For this reason, the results discussed below relate only to the second law of control. It was required to steer the system to the final state $|x_T| \leq 0.1$ in all the examples given below. Small circles denote the positions of the system after 2 s in the phase planes of Figs 1 and 2. The initial points of trajectories correspond to the initial positions of the system.

The phase trajectories of the system $\dot{x}_1 = x_2$, $\dot{x}_2 = u$ are shown in Fig. 1. The system is unstable but a solution $t_2$ exists for all $R_1$ and $R_2$.

The phase trajectories of the system $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 + u$ are shown in Fig. 2. The computations show that the time taken to transfer from any point of the domain of radius $R_1 = 2.1$ into the domain of radius $R_2 = 0.1$ under the action of control (6) is not greater than 6.1. The guaranteed time taken to steer the system from any point of the domain of radius $R_1 = 2.1$ into the specified domain of radius $R_2$ under the action of restricted control (1) is equal to $\pi$.

Consider the pendulum whose point of suspension can move along the horizontal guiding line with a velocity

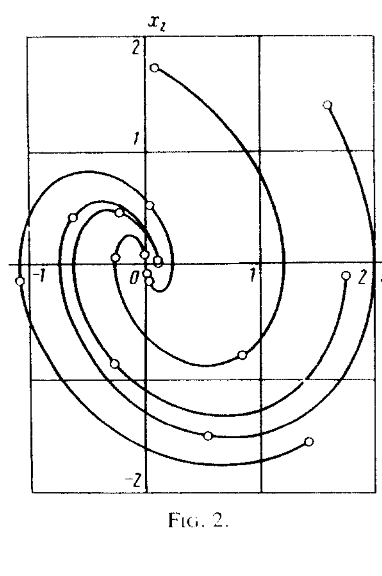


FIG. 2.

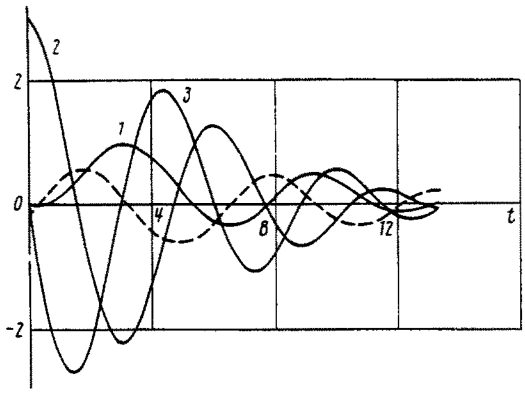


FIG. 3.

that is restricted in magnitude. We will assume that the velocity of the point of suspension can vary within the specified limits almost instantaneously. The equations of motion of such a system are written in the form $\dot{x}_1 = u, \dot{x}_2 = x_3, \dot{x}_3 = -x_3 + u$ [10]. The variable $x_2$ has the meaning of the absolute dimensionless velocity of the weight, and $x_3$ is the angle of deflection of the pendulum from the vertical. Graphs of the variations of the phase coordinates as a function of time are shown in Fig. 3. The numbers on the curves correspond to the index of the space variable. The control function $u(t)$ is shown by the dashed line with the following initial condition: $x_1(0) = x_3(0) = 0, x_2(0) = 3$.

The value of the control fluctuates within the specified limits and does not tend to zero as $t \to T$.

*Remark.* It is interesting to compare the control (6) at $i = 2$, with the control which realizes the optimal synthesis in the problem of steering the dynamical system $\dot{x} = Fx + Gu$ to the origin of coordinates in a fixed time $T$ in the case of a functional that is quadratic with respect to the control. It is well known [7] that the solution of the latter problem yields a control that is linear with respect to the phase coordinates with a feedback matrix which depends on the time needed to complete the motion.

Let system (1) be completely controllable (2), let the corresponding homogeneous system (1) be stable and let $\alpha = 0$ (18). It has been shown [1] that in this case, in order for the condition that control (1) is bounded to be satisfied, the time $T$ of motion must be greater than a certain quadratic function of the radius $R_1$ of domain (3) of the initial positions of the system. It may be obtained from a recurrent equation similar to Eq. (19) for $i = 2$, that the value of $\mu_{2N}$ (19) exceeds a quadratic function $N$ ($N = 1, 2, 3, \ldots$) under the same assumptions. From Eq. (21) it therefore follows that the time $\tau_2$ needed to move into an arbitrary fixed neighbourhood of the origin of coordinates, corresponding to control (6) for $i = 2$, does not exceed a linear function of $R_1$. Therefore, the radius of $R_1$ of the domain of possible initial states of system (1) is sufficiently large, control (6) for $i = 2$ is more effective in its speed of response than the mode of the motion discussed in [1].

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